THE METHOD OF CONCENTRATION COMPACTNESS IN DISPERSIVE HAMILTONIAN EVOLUTION EQUATIONS

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This talk will survey some recent developments in the theory of nonlinear dispersive evolution equations, with emphasis on a qualitative description of the global-in-time dynamics of solutions. We will present the method of concentration compactness which has lead to important advances during the past six years. These results cannot be obtained by perturbative techniques.

In the elliptic setting, concentration compactness was introduced into the calculus of variations by P. L. Lions in the 1980s. The main idea is to exhibit the action of non-compact symmetry groups as the only possible obstruction to compactness. For evolution equations, such ideas were developed by Hajer Bahouri and Patrick Gérard in the late 1990s, with independent work by Frank Merle and Luis Vega at about the same time.

In 2006 Carlos Kenig and Frank Merle introduced a method into nonlinear dispersive equations which allows one to obtain global existence and scattering results for nonlinear evolution equations by means of a contradiction argument based on induction on energy. Roughly speaking, the idea is to show that if the desired result fails, then it does so at a minimal energy. Using concentration compactness, more precisely, a Bahouri-Gérard decomposition, one then constructs a solution at that minimal energy with pre-compact trajectory in the energy space. The final part of the argument, which is typically based on virial-type identities, excludes the existence of such a rigid object.

This method has been applied to different classes of equations. In particular, it was used in joint work with Joachim Krieger to establish global existence and scattering for large data wave maps from 2+1 dimensions into hyperbolic space. We will also present a qualitative description of the flow of focusing nonlinear wave-type equations at energies near the ground state energy. This latter work, joint with Kenji Nakanishi, combines the theory of invariant manifolds — in this case generated by a ground state soliton with a one-dimensional exponential instability in the linearized flow – with some of these concentration-compactness and virial ideas, to exhibit a center-stable manifold as a surface separating an open region of finite-time blowup from another one leading to global existence and scattering to a free wave. A crucial dynamical ingredient in this work is the exclusion of almost homoclinic orbits associated with the ground state.